On the Existence of Equilibria with Entry and Trade in Second-Price Auctions with a Secret Reserve Price

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Abstract

Lovo and Spaenjers (2017) proved that no trade occurs in all equilibria of a second-price auction with a secret reserve price and an entry cost, since no buyer types enter the auction. They assume buyers bid truthfully, which is the unique equilibrium bidding strategy with a positive reserve and at least three buyers. We show that this result does not encompass the case of exactly two buyers, where equilibria with entry and trade may exist when buyers follow non-truthful bidding strategies.

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1 Introduction

In a second-price auction (SPA) with strictly-positive entry costs and a deterministic secret reserve price, Lovo and Spaenjers (2017) (henceforth LS) contend that the unique equilibrium outcome entails no trade. In all equilibria, the seller sets a sufficiently-high reserve that no buyer enters the auction (after learning their type). They prove this claim for the case of truthful bidding (i.e., buyers bid their valuation) in a SPA with asymmetric buyers.

However, additional equilibria characterised by non-truthful bidding may emerge in a SPA with a public reserve and no entry cost. Blume and Heidhues (2004) (henceforth BH) characterised all of these equilibria, noting that (i) truthful bidding is the unique equilibrium with $N \geq 3$ buyers, and (ii) additional equilibria arise when $N = 2$. Regarding the latter, LS provided an example to show that their no-trade result extends to cover the case of non-truthful bidding with two buyers.

In this note, we revisit this setting (SPA with entry costs and a secret reserve) but focus on the case of exactly two buyers. We show that equilibria with entry and non-truthful bidding (consistent with BH's characterisation) may exist in a SPA with strictly-positive entry costs and a deterministic secret reserve when $N=2$. Therefore, no-entry/no-trade is not the unique equilibrium outcome of the SPA.

The key difference between our finding and LS's unique no-trade outcome is due to the likelihood of bidding at the reserve under each bidding strategy. With truthful bidding, profitable deviations from the anticipated reserve price always exist for reserves low enough to induce entry (since bidding at the reserve is relatively unlikely). However, under non-truthful bidding, sellers face a trade-off: deviating from the anticipated reserve price raises revenue in some states but triggers a buy-in (i.e., no sale) in others (since bidding at the reserve is relatively likely).

By quantifying this trade-off, we determine when no profitable deviation (from a reserve price that induces entry) exists, and thus characterise when an equilibrium with trade exists. We then analyse how existence depends on model primitives via comparative statics, and use numerical examples to demonstrate the incidence of trade and the role of buyer asymmetry.

2 Setup

There are two risk-neutral buyers $i \in \{1,2\}$ with independent private values $v_i \sim F_i$ supported on $[0,1]$ with continuous, strictly-positive densities. Entering the auction costs $c \in (0,1)$ and is sunk. The seller is risk neutral and has an outside option that yields 0 payoff. The mechanism is a sealed-bid second-price auction (SPA) with a secret reserve $x \in [0,1]$ chosen by the seller.

Consider a candidate reserve $r \in [0,1]$. If this reserve is secret, it is not observed at the time of entry or bidding, hence buyers must anticipate the level of the reserve. For a candidate reserve to be consistent with a Perfect Bayesian Equilibrium (PBE), it must be correctly anticipated by buyers for both entry and bidding strategies. If a seller's best response to optimal entry and bidding strategies based on an anticipated/candidate reserve r is to set the secret reserve at $x = r$ (i.e., no incentive to deviate), then it constitutes an equilibrium. Thus, despite being secret and unknown to buyers, in any PBE it is correctly inferred.

Let w_i denote the entry threshold, where buyers make entry decisions after learning their type, so that buyer i enters if and only if $v_i \geq w_i$.[1] Following entry, consider a two-bidder equilibrium with a "flat" segment on $[w_i, W]$:

$$b_1(v_1) = \begin{cases} r, & v_1 \in [w_1, W], \\ v_1, & v_1 > W, \end{cases} \qquad b_2(v_2) = \begin{cases} W, & v_2 \in [w_2, W], \\ v_2, & v_2 > W. \end{cases}$$

The upper endpoint of the flat segment W is arbitrary, however, it must be feasible given the entry cost. Therefore, $W \in (W_{\min}, 1)$, where $W_{\min} := \max\{w_1, w_2\}$.

Given the post-entry bidding strategies above, the entry thresholds (w_1, w_2) solving the zero-profit conditions are:

$$c \;=\; F_2(w_2)\,(w_1 - r), \qquad c \;=\; F_1(W)\,(w_2 - r). \tag{1}$$

We note that all of the above assumptions regarding F_i, c, r, and x are identical to those made by LS (except we restrict our attention to $N = 2$).[2] The above bidding strategy also aligns with that specified in their example, since it is consistent with the two-bidder Vickrey literature, where "gaps" above the reserve may arise in equilibrium (Plum, 1992; Blume and Heidhues, 2001, 2004).

[1] See Moreno and Wooders (2017) for a similar setting, i.e., second-price auction with endogenous entry and a secret reserve, where buyers learn their type after entry.

[2] One point of departure, however, is the entry threshold for the buyer bidding at the top of the "flat", i.e., at W (buyer 2, in our setup). LS specify a symmetric w_i, obtained by solving $F_{-i}(w_{-i})(w_i - r) = c$ for both buyers. When $N = 2$, this yields a conservative entry cutoff for buyer 2, given the above bidding strategies, since they win the post-entry bidding for any $v_1 \leq W$.

3 Existence of equilibria with entry and trade

We ask whether the seller, expecting (w_1, w_2) and the flat-segment bidding in Section 2, can profitably raise the secret reserve to any $x \in (r, 1]$. To this end, let $R(x, r)$ denote the seller's expected revenue when buyers best-respond (i.e., enter and bid) to the anticipated reserve r, but the seller secretly sets the reserve x. Henceforth, we shall concisely refer to $x > r$ as a "secret raise", i.e., an upwards deviation from the anticipated reserve. In this environment, an equilibrium is characterised by a reserve r^* that satisfies $r^* \in \arg\max_{x\in[0,1]} R(x, r^*)$.

First, we define a function $\Delta(x)$ that gives the expected payoff of a secret raise, i.e., the change in expected revenue when setting the secret reserve at x (relative to r). Next, we show that $\Delta(x)$ is increasing along the "flat", i.e., for secret raises $x \in (r, W]$, hence it attains its maximum at W. Finally, we show when raises beyond the flat, i.e., $x \in (W, 1]$, cannot improve upon $\Delta(W)$; thus, we can characterise the seller's incentive to deviate in terms of $\Delta(W)$.

Lemma 1 (Expected payoff of a secret raise)**.** *For every $x \in (r, W]$,*

$$\Delta(x) \; := \; R(x,r) - R(r,r) \; = \; P^S(x-r) - P^B r, \tag{2}$$

where

$$\begin{aligned} \Pr\{sale\} \; &= \; P^S \; := \; F_1(W)\big[1 - F_2(w_2)\big] \; + \; \big[1 - F_1(W)\big]F_2(w_2), \\ \Pr\{buy\text{-}in\} \; &= \; P^B \; := \; \big[F_1(W) - F_1(w_1)\big]\, F_2(w_2). \end{aligned}$$

Note that P^S in (2) is defined to measure the probability of all events where a sale occurs *and the seller's payoff rises* (due to the secret raise). In particular, it does not include any events where a sale occurs but revenue does not rise relative to the anticipated reserve. Therefore, P^S is the slope of $\Delta(x)$, it is strictly positive, and thus $\Delta(x)$ is strictly increasing on $(r, W]$. Additionally, $\Delta(x)$ has a downward jump at $x = r^+$ equal to the buy-in loss, $\Delta(r^+) - \Delta(r) = -P^B r < 0$ (where $\Delta(r) = 0$, by definition).

Proposition 1 (Break-even threshold)**.** *Let $\hat{x} := r\big(1 + \frac{P^B}{P^S}\big)$. Then, for $x \in (r, W]$,*

$$\Delta(x) \le 0 \iff x \le \hat{x}, \qquad \Delta(x) > 0 \iff x > \hat{x}.$$

If $\hat{x} \ge W$, then $\Delta(x) \le 0$ for every $x \in (r, W]$, so no secret raise is profitable. Additionally, the largest and most profitable secret raise (i.e., at $x = W$) is unprofitable if and only if

$$\Delta(W) \le 0 \iff \frac{W - r}{r} \; \le \; \frac{P^B}{P^S}. \tag{3}$$

So far we have restricted our consideration of secret raises to along the "flat", $(r, W]$, where $W < 1$, in line with the example of LS.[3] However, since our goal is to identify conditions for the existence of equilibria with trade, we must ensure that no profitable deviation exists for any $x > r$. Thus, we must also consider deviations beyond W, i.e., $x \in (W, 1]$.

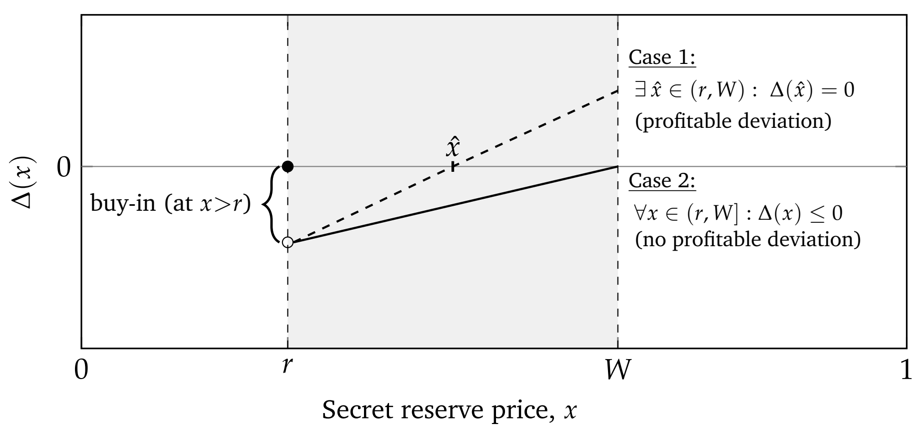
The following Lemma shows, however, that the condition in Proposition 1 (i.e., $\Delta(W) \leq 0$) is sufficient rule out the existence of any profitable deviations, hence we do not need to extend our search for profitable deviations beyond W.

Lemma 2 (Global best secret raise)**.** *Fix the entry cutoffs (w_1, w_2) and equilibrium bidding strategies induced by $c \in (0,1)$, $r \in [0,1]$, and $W \in (W_{\min}, 1)$. For $x \geq W$, define $G_i(x) := \Pr\{b_i(v_i) \geq x\}$. If $G_i(x) + xG_i'(x) \leq 0$ for all $x \in [W, 1]$ and $i \in \{1,2\}$, then $R(\cdot, r)$ is nonincreasing on $[W, 1]$.[4] Therefore, given that $R(\cdot, r)$ is strictly increasing on $(r, W]$,*

$$W \in \arg\max_{x \in (r,1]} R(x, r).$$

We now proceed to illustrate the above results characterising existence in Figure 1. The diagram depicts the two possible cases of $\Delta(x)$, in terms of $\Delta(W)$ and $\hat{x}$ (defined in Proposition 1), either of which equivalently determine whether an equilibrium with trade exists.

Figure 1: Expected payoff from a secret raise, $\Delta(x)$



Case 1 corresponds to the no-trade result of LS, since $\hat{x}$ is interior (i.e., $r < \hat{x} < W$), or, equivalently, $\Delta(W) > 0$. Case 2 corresponds to the existence of an equilibrium with trade, since $\hat{x} \geq W$, or, equivalently, $\Delta(W) \leq 0$ (i.e., no profitable deviation exists for the seller).

[3]In LS's example with $N = 2$, which was characterised by bidding profiles with "flats" (described by BH), it was enough for their purposes to restrict their analysis to deviations on the interval $(r, W]$. This is because finding any such profitable deviation in that interval is sufficient to conclude an equilibrium with trade does not exist.

[4]Equivalently, whenever $G_i(x) > 0$, assume the reverse hazard $\rho_i(x) := -G_i'(x)/G_i(x) \geq 1/x$.

Corollary 1 (Existence of an equilibrium with trade). *If (3) holds, i.e., $\Delta(W) \le 0$ (or, equivalently, $\frac{W-r}{r} \le \frac{P^B}{P^S}$), then every secret raise $x \in (r,1]$ is weakly unprofitable; thus, setting the secret reserve at r constitutes an equilibrium with entry and trade.*

The existence condition from Proposition 1, $\frac{W-r}{r} \le \frac{P^B}{P^S}$, and the tail/reverse-hazard condition from Lemma 2, $\rho_i(x) \ge 1/x$, are met for common functional forms of $F_i(\cdot)$ and a variety of parameterisations of c, r, and W. We provide several numerical examples in Section 3.2.

3.1 Comparative statics

Given entry cutoffs (w_1, w_2) determined by (1), an increase in c or r raises w_2 strictly (for fixed W). The response of w_1 includes a direct positive effect (via r or c) and an indirect negative effect through w_2 (because $F_2(w_2)$ rises); in typical benchmarks w_1 also rises, but this is not required for our conclusions. In any case, the feasibility boundary $W_{\min}$ moves weakly to the right as c or r increase, because w_2 rises strictly.

Fixing W and evaluating $\Delta(x)$ at its maximiser, $x = W$, we note three forces affect $\Delta(W)$:

(i) *Interval length (i.e., domain effect).* A higher r shrinks $W - r$ pointwise. A higher c tightens feasibility by pushing $W_{\min}$ right; this reduces the *attainable* $W - r$ only insofar as W must move with $W_{\min}$ to remain admissible.

(ii) *Buy-in term.* As r rises, the loss term $(-P^B r)$ becomes larger in magnitude, while $P^B = [F_1(W) - F_1(w_1)]F_2(w_2)$ typically falls because w_1, w_2 increase. The net effect on $\Delta(W)$ is generally negative, but its sign ultimately depends on the relative effect sizes.

(iii) *Slope.* Since $\frac{\partial P^S}{\partial w_2} = f_2(w_2)\big(1 - 2F_1(W)\big)$, whenever $F_1(W) \ge \frac{1}{2}$, increases in w_2 (triggered by higher c or r) reduce P^S and thereby reduce $\Delta(W)$ at a fixed W.

Thus, as c or r rise, the admissible set $\{W \ge W_{\min}(c,r)\}$ shrinks, and—in standard cases with $F_1(W) \ge \frac{1}{2}$ (e.g., symmetric uniform near moderate W)—the pointwise effects (ii)–(iii) reinforce the domain effect, so the attainable $\Delta(W)$ weakly decreases. Profitable secret raises are therefore less likely. If instead $F_1(W) < \frac{1}{2}$, the slope effect (iii) works in the opposite direction and can partially offset (i)–(ii), so signs can be ambiguous.

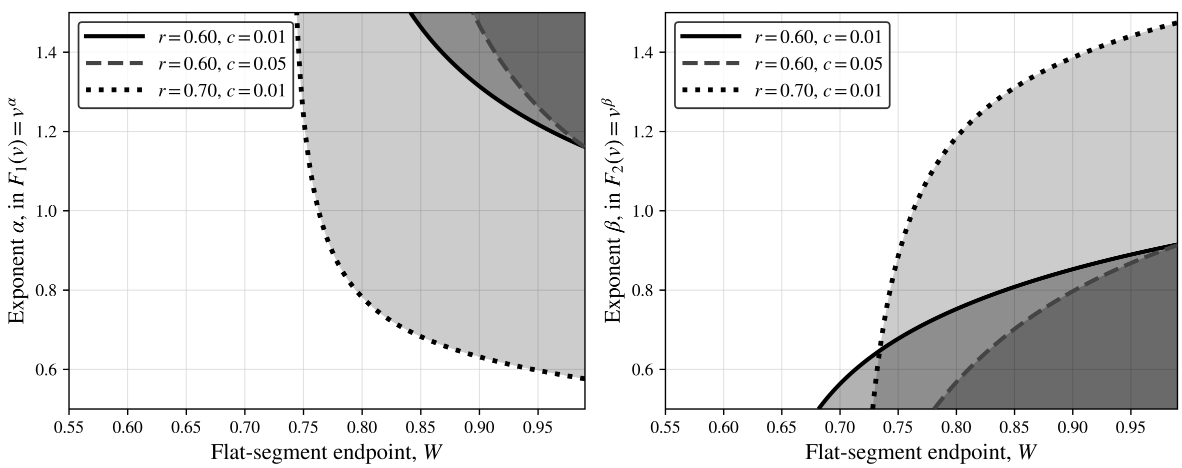
3.2 Numerical example

Consider two asymmetric buyers with independent private values on $[0,1]$ distributed $F_1(v) = v^\alpha$ and $F_2(v) = v^\beta$, with $\alpha, \beta \in [\frac{1}{2}, \frac{3}{2}]$. We proceed by setting either α or β equal to 1 (i.e., so F_1 or F_2 is uniformly distributed) and varying the other parameter to investigate the impact of buyer asymmetry on the existence of equilibria with trade.

Figure 2 depicts when profitable deviations do not exist (and hence equilibria with trade exist) for particular values of the reserve r, entry costs c, preferences α and β, and the flat-segment endpoint W. It restricts attention to the maximum profitable deviation, i.e., $x = W$, and plots the level sets of $\Delta(W) = 0$ for varying (W, α). In particular, the shaded regions reveal pairs of W and preferences (α or β) where an equilibrium with entry and trade exists.

The left panel of Figure 2 sets $\beta = 1$ so that $F_2(v_2) = v_2$, while the right panel sets $\alpha = 1$ so that $F_1(v_1) = v_1$. Therefore, along the horizontal line at $\alpha = \beta = 1$ the figures are identical, but for all other pairs of $(W, \cdot)$ they differ due to the asymmetric bidding strategies specified in Section 2. Above (below) $\alpha = \beta = 1$, the respective buyer is relatively "strong" ("weak") since, e.g., raising β above 1, decreases $F_2(\cdot)$ relative to the uniform distribution. Finally, r and c are chosen in the particular examples to illustrate the effects described in Section 3.1.

Figure 2: Level sets of $\Delta(W) = 0$



Note: The shaded regions represent non-positive values of $\Delta(W)$, i.e., its lower contour set: $\{(W, \alpha) \mid \Delta(W) \leq 0\}$. In these regions, there exists an equilibrium with entry and trade.

First, fixing $r = 0.6$ and raising c from 0.01 to 0.05 indeed tightens feasibility, leading to a rightward compression of the shaded region as the feasible set $W_{\min}$ shifts to the right (in both panels). Second, fixing $c = 0.01$ and raising r from 0.60 to 0.70 generally enlarges the set of equilibria with trade (subject to the feasible set shrinking).

There is also a noticeable asymmetry across panels in terms of the range of W encompassed in the darker-shaded regions. This can be explained by the different impact on the feasible set when each bidder becomes globally stronger, i.e., as $F_i(\cdot)$ falls. A higher value of α "strengthens" buyer 1 (bidding at the flat r), while a lower value of β "weakens" buyer 2 (bidding at the flat W).

For buyer 1: as $F_1(\cdot)$ falls, w_2 rises and w_1 falls, pushing the feasibility boundary rightward; this domain effect uniformly works against profitable secret raises. For buyer 2: as $F_2(\cdot)$ rises,

$w_1 = r + c/F_2(w_2)$ falls but $w_2 = r + c/F_1(W)$ remains fixed, therefore the feasibility boundary remains fixed. Additionally, for high W, as in the figure, $F_1(W) > \frac{1}{2}$, so $\partial P^S/\partial F_i$ falls, reducing $\Delta(W)$; but the main driving force behind the difference across panels is the asymmetry in the rightward-shift of the feasible set, which doesn't depend on the functional forms of $F_i(\cdot)$.[5]

4 Concluding remarks

We studied the incentives of a seller setting a secret reserve price in a second-price auction with entry costs and $N = 2$ buyers that know their type before entry. In stark contrast to the case of $N \geq 3$, where buyers pursue truthful bidding strategies and thus no equilibria with trade exist (Lovo and Spaenjers, 2017), we showed that equilibria with entry and trade may exist when buyers follow non-truthful bidding strategies.

Equilibria with non-truthful bidding strategies are typically not a focus of studies on Vickrey auctions; e.g., Plum (1992) described them as "pathological", noting that they are dominated by truth-telling. On the other hand, Blume and Heidhues (2001) provide an interesting example demonstrating that non-truthful equilibria can payoff-dominate truthful equilibria in repeated games with bid rotation (for the bidders).

Our findings provide an additional justification for studying these strategies: whenever a trade equilibrium exists in our setting with $N = 2$ buyers, the resulting outcome strictly payoff-dominates every truthful-bidding equilibrium outcome (all of which are no-entry) for the seller; for buyers, conditional on trade, the realised winner earns strictly-positive surplus.

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[5]The general effect on $\Delta(W)$ is *a priori* ambiguous for both buyers. For buyer 1, the sign of the change in $\Delta(W)$ depends on whether w_2 and $F_1(W)$ lie above or below $\frac{1}{2}$. For buyer 2, the change in $-rP^B$ reduces $\Delta(W)$, since $P^B = [F_1(W) - F_1(w_1)]F_2(w_2)$ increases, but the effect of P^S depends on whether $F_1(W)$ lies above or below $\frac{1}{2}$.

Appendix

Lemma 1 (Expected payoff of a secret raise). *For every $x \in (r, W]$,*

$$\Delta(x) \;:=\; R(x,r) - R(r,r) \;=\; P^S\big(x - r\big) - P^B r, \tag{2}$$

where

$$\begin{aligned} \Pr\{sale\} \;&=\; P^S \;:=\; F_1(W)\big[1 - F_2(w_2)\big] \;+\; \big[1 - F_1(W)\big]F_2(w_2), \\ \Pr\{buy\text{-}in\} \;&=\; P^B \;:=\; \big[F_1(W) - F_1(w_1)\big]\,F_2(w_2). \end{aligned}$$

Proof. Under the BH profile at reserve r, any sale at price r occurs in one of the mutually exclusive cases:

- Only bidder 2 enters:
  (a) $v_2 \in [w_2, W]$ (then $b_2 = W \geq r$);
  (b) $v_2 > W$ (then $b_2 = v_2 > W \geq r$).
- Both enter, bidder 1 flat:
  (c) $v_1 \in [w_1, W]$, $v_2 \in [w_2, W]$ (bids r and W);
  (d) $v_1 \in [w_1, W]$, $v_2 > W$ (bids r and $v_2 > W$).
- Only bidder 1 enters:
  (e) $v_1 \in [w_1, W]$ (then $b_1 = r$);
  (f) $v_1 > W$ (then $b_1 = v_1 > W$).

By independence, the corresponding probabilities are

$$\begin{aligned} &\Pr(\mathrm{a}) = F_1(w_1)\,[F_2(W) - F_2(w_2)], && \Pr(\mathrm{b}) = F_1(w_1)\,[1 - F_2(W)], \\ &\Pr(\mathrm{c}) = [F_1(W) - F_1(w_1)]\,[F_2(W) - F_2(w_2)], && \Pr(\mathrm{d}) = [F_1(W) - F_1(w_1)]\,[1 - F_2(W)], \\ &\Pr(\mathrm{e}) = [F_1(W) - F_1(w_1)]\,F_2(w_2), && \Pr(\mathrm{f}) = [1 - F_1(W)]\,F_2(w_2). \end{aligned}$$

On $x \in (r, W]$:

- in (a)–(d),(f) the sale persists and the price increases by $(x - r)$;
- in (e) the sale at r is lost (loss $= r$).

Summing (a)–(d) gives $F_1(W)\,[1 - F_2(w_2)]$; adding (f) yields P^S. The buy-in probability is $\Pr(\mathrm{e}) = P^B$. Linearity of expectation gives (2). □

Proposition 1 (Break-even threshold). *Let* $\hat{x} := r\big(1 + \frac{P^B}{P^S}\big)$*. Then, for* $x \in (r, W]$*,*

$$\Delta(x) \le 0 \iff x \le \hat{x}, \qquad \Delta(x) > 0 \iff x > \hat{x}.$$

If $\hat{x} \ge W$*, then* $\Delta(x) \le 0$ *for every* $x \in (r, W]$*, so no secret raise is profitable. Additionally, the largest and most profitable secret raise (i.e., at* $x = W$*) is unprofitable if and only if*

$$\Delta(W) \le 0 \iff \frac{W - r}{r} \;\le\; \frac{P^B}{P^S}. \tag{3}$$

Proof. By Lemma 1, for $x \in (r, W]$ we have $\Delta(x) = P^S\,(x - r) - r\,P^B$, with $P^S > 0$ independent of x on $(r, W]$. Hence $\Delta(\cdot)$ is strictly increasing on $(r, W]$ and attains its maximum at $x = W$. Solving $\Delta(x) = 0$ gives

$$x \;=\; r\left(1 + \tfrac{P^B}{P^S}\right) \;=:\; \hat{x}.$$

Therefore, $\Delta(x) \le 0$ iff $x \le \hat{x}$, and $\Delta(x) > 0$ iff $x > \hat{x}$. In particular, $\Delta(W) \le 0$ iff $W \le \hat{x}$, i.e.,

$$\Delta(W) \le 0 \iff \frac{W - r}{r} \;\le\; \frac{P^B}{P^S}.$$

Since $\Delta(\cdot)$ is increasing on $(r, W]$, $\Delta(W) \le 0$ is equivalent to $\Delta(x) \le 0$ for all $x \in (r, W]$. □

Lemma 2 (Global best secret raise). *Fix the entry cutoffs* (w_1, w_2) *and equilibrium bidding strategies induced by* $c \in (0, 1)$*,* $r \in [0, 1]$*, and* $W \in (W_{\min}, 1)$*. For* $x \ge W$*, define* $G_i(x) :=$ $\Pr\{b_i(v_i) \ge x\}$*. If* $G_i(x) + xG_i'(x) \le 0$ *for all* $x \in [W, 1]$ *and* $i \in \{1, 2\}$*, then* $R(\cdot, r)$ *is nonincreasing on* $[W, 1]$*.*[6] *Therefore, given that* $R(\cdot, r)$ *is strictly increasing on* $(r, W]$*,*

$$W \in \arg\max_{x \in (r,1]} R(x, r).$$

Proof. Let $B_{(1)} \ge B_{(2)}$ be the order statistics of the two bids, i.e., $B_{(1)} := \max\{b_1, b_2\}$ and $B_{(2)} := \min\{b_1, b_2\}$. As in Section 3,

$$R(x, r) = x\,\Pr\{B_{(1)} \ge x > B_{(2)}\} + \mathbb{E}\Big[B_{(2)}\,\mathbf{1}\{B_{(2)} \ge x\}\Big]. \tag{4}$$

On $x \in (r, W]$, the event sets in (4) do not change with x under the BH post-entry bidding profile (i.e., raising x within the flat does not alter who buys), hence $R(\cdot, r)$ is linear with strictly-positive slope $P^S > 0$. Therefore, R is strictly increasing on $(r, W]$. For $x \ge W$, increasing x has two effects: (i) it reduces the probability of sale by converting any state with $B_{(1)} \in (W, x)$

[6] Equivalently, whenever $G_i(x) > 0$, assume the reverse hazard $\rho_i(x) := -G_i'(x)/G_i(x) \ge 1/x$.

into a buy-in, and (ii) it raises the payment by Δx in the shrinking set $\{B_{(1)} \geq x > B_{(2)}\}$.

Define $G_i(x) := \Pr\{b_i(v_i) \geq x\} = 1 - \Pr\{b_i(v_i) < x\}$, for $x \geq W$. Because $b_1(v_1)$ and $b_2(v_2)$ are independent,

$$\begin{aligned}\Pr\{B_{(1)} \geq x > B_{(2)}\} &= \Pr\{b_1 \geq x,\ b_2 < x\} + \Pr\{b_2 \geq x,\ b_1 < x\} \\ &= G_1(x)\big(1 - G_2(x)\big) + \big(1 - G_1(x)\big)G_2(x) \\ &= G_1(x) + G_2(x) - 2G_1(x)G_2(x). \end{aligned} \tag{5}$$

Also $\Pr\{B_{(2)} \geq x\} = \Pr\{b_1 \geq x,\ b_2 \geq x\} = G_1(x)G_2(x)$. Using the standard tail derivative

$$\frac{d}{dx}\,\mathbb{E}\big[B_{(2)}\,\mathbf{1}\{B_{(2)} \geq x\}\big] = x\,\frac{d}{dx}\Pr\{B_{(2)} \geq x\} \quad \text{(at continuity points of } B_{(2)}\text{)}.$$

Differentiating (4) gives

$$\begin{aligned}R'(x,r) &= \Pr\{B_{(1)} \geq x > B_{(2)}\} + x\,\frac{d}{dx}\Pr\{B_{(1)} \geq x > B_{(2)}\} + x\,\frac{d}{dx}\Pr\{B_{(2)} \geq x\} \\ &= \ \big(1 - G_2(x)\big)\,\big[G_1(x) + x\,G_1'(x)\big] \ + \ \big(1 - G_1(x)\big)\,\big[G_2(x) + x\,G_2'(x)\big], \end{aligned} \tag{6}$$

where the final equality follows from substituting (5).

Note that the weights $1 - G_i(x) = \Pr\{b_i < x\} \geq 0$ for all x. Hence, a sufficient condition for $R'(x,r) \leq 0$ on $[W,1]$ is

$$G_i(x) + x\,G_i'(x) \ \leq \ 0 \qquad \text{for all } x \in [W,1],\ i = 1,2. \tag{7}$$

Equivalently, with the reverse hazard $\rho_i(x) := -\,G_i'(x)/G_i(x)$,

$$\frac{d}{dx}\big[x\,G_i(x)\big] = G_i(x)\big(1 - x\,\rho_i(x)\big) \leq 0 \quad \Longleftrightarrow \quad \rho_i(x) \geq \frac{1}{x}.$$

Thus, the pointwise bound $\rho_i(x) \geq 1/x$ on $[W,1]$ for each i implies (7), which, by (6) is sufficient for $R'(x,r) \leq 0$ for all $x \geq W$.

Given that $R(\cdot,r)$ is strictly increasing on $(r,W]$ (Lemma 1) and weakly decreasing on $[W,1]$ under (7), any maximiser over $x > r$ must lie in $[W,1]$ and yield a value no greater than at W. Thus $W \in \arg\max_{x\in(r,1]} R(x,r)$. The final claim (Corollary 1) follows since $\Delta(W) \leq 0$ implies $R(W,r) \leq R(r,r)$, and thus $R(x,r) \leq R(W,r) \leq R(r,r)$ for all $x \geq W$. □

Numerical example (in Section 3.2): (*Uniform*) $F(x) = x$, so $G(x) = 1 - x$, and hence $G + xG' = 1 - 2x \leq 0$ for $x \geq \frac{1}{2}$. Thus any $W \geq \frac{1}{2}$ satisfies (7); (*Power*) $F(x) = x^{\alpha}$, so $G(x) = 1 - x^{\alpha}$, and hence $G + xG' = 1 - (1+\alpha)x^{\alpha} \leq 0$ for $x \geq (1/(1+\alpha))^{1/\alpha}$; any such W suffices.